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LONGITUDINAL COUPLING IMPEDANCE OF PICKUP
PLATES WITH TERMINATIONS AT BOTH ENDS

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The purpose of this note is to compute the longitudinal impedance of a pickup plate with terminations at both ends[1]. The plate has length ℓ, characteristic impedance Z_0 and azimuthal half angle ϕ_0 (Figure 1). Each termination carries an impedance Z_T.

1. Transmission line equations.

When a linear charge disturbance

$$\lambda_1 = \lambda_{10}\, e^{j(\omega t - kz)} \tag{1}$$

develops in the beam, due to charge conservation, the disturbance current is

$$I_1 = I_{10}\, e^{j(\omega t - kz)}, \tag{2}$$

with

$$I_{10} = \lambda_{10}\beta_W c \tag{3}$$

where $k = n/R$ defines the longitudinal mode number n and ω the angular velocity of the disturbance; R is the radius of the machine. $\beta_W c = \omega/k$ is the linear velocity of the disturbance. z is the direction of motion of the particles. This disturbance will induce a charge density σ_1 and current density J_1 on the pickup plate:

$$\sigma_1 = -\frac{\lambda_1}{2\pi b}\, g^{long}, \tag{4}$$

$$J_1 = -\frac{\lambda_1}{2\pi b}\, g^{long}, \tag{5}$$

where $q = k\sqrt{1-\beta_W^2}$; a and b are the radii of the beam and the pipe respectively; I_0 and I_1 are modified Bessel functions of order 0 and 1 respectively. Note that, as $q \to 0$, $g^{long} \to 1$.

The pickup plate and ground form a transmission line. The line equations for the supplementary scalar potential V_1 and longitudinal potential A_1 (set to zero for ground) produced by the plate are

$$\frac{1}{c}\frac{\partial V_1}{\partial t} + \frac{\partial A_1}{\partial z} = 0, \tag{7}$$

$$\frac{\partial V_1}{\partial z} + \frac{1}{c}\frac{\partial A_1}{\partial t} = 0. \tag{8}$$

Equation (7) expresses the charge conservation law and is homogeneous because the linear velocity and the phase velocity of a longitudinal disturbance are exactly equal. Equation (8) is homogeneous because the plate is assumed to be a perfect conductor.

The most general solution of the transmission equation is

$$\begin{Bmatrix} V_1(z,t) \\ A_1(z,t) \end{Bmatrix} = (ae^{j\frac{\omega}{c}(z_s-z)} \pm be^{-j\frac{\omega}{c}(z_s-z)})e^{j\omega t} \tag{9}$$

subject to the boundary conditions that the currents at both ends of the plate extending from $z=z_o$ to $z_s+\ell$ must be zero:

$$\frac{A_1(z_s,t)}{Z_o} = -\frac{V_1(z_s,t)}{Z_r} - 2\phi_o bJ_1(z_s,t), \tag{10}$$

$$\frac{A_1(z_s+\ell,t)}{Z_o} = \frac{V_1(z_s+\ell,t)}{Z_r} - 2\phi_o bJ_1(z_s+\ell,t). \tag{11}$$

If we match $Z_o = Z_T$, we get

$$a = -\phi_o bZ_oJ_s, \tag{12}$$

$$b = \phi_o bZ_oJ_s \; e^{-j\frac{\omega}{c}\ell - jk\ell}, \tag{13}$$

with

$$J_s = -\frac{I_{10}}{2\pi b} g^{long} e^{-jkz_s}. \tag{14}$$

2. Longitudinal Impedance

Elsewhere (not necessarily on the plate), the supplementary potentials, since obeying Eqs. (7) and (8), can be written as

$$\begin{Bmatrix} V_1 \ (\rho, \phi, z, t) \\ A_1 \ (\rho, \phi, z, t) \end{Bmatrix} = \sum_{p,h} \phi_p \begin{Bmatrix} V_h \\ A_h \end{Bmatrix} \frac{I_p(q\rho)}{I_p(qb)} \cos p\phi \, e^{j(\omega t - \frac{h}{R} z)} .$$

As all modes $p \neq 0$ and $h \neq n$ are orthogonal to the fundamental mode of the disturbance, the forces produced by these modes are ineffective over a complete turn in the machine. We therefore retain only $h=n$ and $p=0$. Defining

$$\Phi_o \equiv \frac{2\phi_o b}{2\pi} \int_{-\phi_o}^{\phi_o} d\phi = \frac{2b\phi_o^2}{\pi} ,$$

we get

$$\begin{Bmatrix} V_n \\ A_n \end{Bmatrix} e^{j\omega t} = \frac{1}{2\pi R} \frac{1}{2b\phi_o} \int_{z_s}^{z_s+\ell} dz \, e^{j\frac{n}{R}z} \begin{Bmatrix} V_1 \\ A_1 \end{Bmatrix}_{\rho=0}$$

$$= \frac{\ell}{4\pi R} \frac{Z_o J_s e^{jkz_s}}{j\theta(1-\beta_w^2)} \begin{Bmatrix} \beta_w C_1 + C_2 \\ \beta_w C_2 + C_1 \end{Bmatrix} e^{j\omega t} \tag{15}$$

with

$$C_1 = -\sin 2\Phi \sin 2\theta - j \sin 2\theta \cos 2\Phi,$$

$$C_2 = 1 - \cos 2\Phi \cos 2\theta + j \cos 2\theta \sin 2\Phi, \tag{16}$$

$$2\theta = k\ell,$$

$$2\Phi = \frac{\omega}{c} \ell. \tag{17}$$

We note that Eq. (15) is independent of z_s, the position of the plate along the beam pipe. Thus for M identical plates, we just multiply Eq. (15) by M. At the center of the beam, $\rho=0$, the supplementary longitudinal electric field due to the M plates is

$$E_z(\rho=0) = -\frac{\partial V_1}{\partial z} - \frac{1}{c} A_1$$

$$= jk \frac{V_n - \beta_w A_n}{I_o(qb)} \Phi_o \, e^{j(\omega t - kz)}$$

$$= \frac{M}{2\pi R} \left(\frac{2b\phi_o^2}{\pi}\right) \frac{Z_o J_1}{I_o(qb)} \, C_2. \tag{18}$$

The potential seen by the beam in one resolution is

$$U_s = 2\pi R \, E_z(\rho = 0). \tag{19}$$

Therefore, the longitudinal impedance due to the plate is

$$Z_L = -\frac{U_s}{I_1}$$

$$= M\left(\frac{\phi_o}{\pi}\right)^2 Z_o \frac{g^{long}}{I_o(qb)} \quad C_2. \tag{20}$$

We now let $\beta_w \to 1$, then

$$g^{long}/I_o(qb) \to 1,$$

$$\theta \to \Phi,$$

$$C_2 \to \sin^2 2\Phi + j \sin 2\Phi \cos 2\Phi.$$

Thus

$$Z_L = M\left(\frac{\phi_o}{\pi}\right)^2 Z_o \left(\sin^2 2\Phi + j \sin 2\Phi \cos 2\Phi\right), \tag{21}$$

agreeing exactly with Shafer's result[2]. When the wavelength of the disturbance is long compared with R/n, we get

$$Z_L/n = jM\left(\frac{\phi_o}{\pi}\right)^2 Z_o \frac{\ell}{R} \tag{22}$$

which is inductive.

3. Voltage along a pickup plate

Substituting (12) and (13) into Eq. (9), we get the voltage along a plate

$$V_1\,(z,t) = -\phi_o b Z_o J_s \left[e^{j\frac{\omega}{c}(z_s-z)} - e^{-j\frac{\omega}{c}(z_s-z+2\ell)}\right] e^{j\omega t} \tag{23}$$

assuming β_w=1. Obviously $V_1(z_s+\ell,t)$=0, i.e., the downstream end of the plate is floating, a result predicted by Shafer[2]. Thus the donwstream termination can be removed without affecting the whole system. As a result, our result can be compared with that obtained by Ruggiero[1] for pickup plates with only one termination situated at a distance $\frac{\ell}{2}$ (1+δ) from the upstream end (δ ranges from -1 to 1). His value of longitudinal force per unit charge at the center of the beam is

$$F^{long} = E_z(\rho=0) = j8\,\frac{M\ell}{2\pi R}\left(\frac{\phi_o^2}{2\pi^2}\right)\left(\frac{g^{long}}{I_o(qb)}\right)\left(P^{long}\right)\frac{\lambda_1}{\ell C}, \tag{24}$$

where $C=(cZ_o)^{-1}$ is the capacitance per unit length of the plate with respect to ground and

$$P^{long} = \frac{\beta w}{2} \frac{2jr(\cos 2\Phi - \cos 2\theta) - \sin 2\Phi}{\cos 2\delta\Phi + \cos 2\Phi + 2jr \sin 2\Phi}, \tag{25}$$

with $r=Z_r/Z_o$. When the impedances match ($r=1$), the termination is at the upstream end of the plate ($\delta=-1$), and $\beta_w=1$, we get

$$P^{long} = -\frac{1}{4}(\sin 2\Phi \cos 2\Phi - j \sin^2 2\Phi). \tag{26}$$

Using Eqs. (2) and (3), (19) and (20), we arrive at

$$Z_L = M \left(\frac{\phi_o}{\pi}\right)^2 Z_o (\sin^2 2\Phi + j \sin 2\Phi \cos 2\Phi). \tag{27}$$

which agrees with Eq. (21).

From Eq. (23), the voltage at the upstream end of a plate is

$$V_1(z_s,t) = \frac{\phi_o b I_{10} Z_o}{2\pi b} (\sin^2 2\Phi + j \sin 2\Phi \cos 2\Phi) e^{j(\omega t - \frac{n}{R} z_s)}. \tag{28}$$

Thus, the average power consumed for M plates is

$$\langle P\rangle = \frac{1}{2} \frac{|V_1(z_s,t)|^2}{\mathrm{Re} Z_T} = \frac{1}{2} M \left(\frac{\phi_o}{\pi}\right)^2 Z_o |I_{10}|^2 \sin^2 2\Phi \tag{29}$$

which equals, as it should, $\frac{1}{2} \mathrm{Re} Z_L |I_{10}|^2$, the average power lost by the beam.

4. Arbitrary Z_T

Matching boundary conditions (10) and (11), line equations (7) and (8) lead to a supplementary longitudinal impedance (due to M plates) of

$$Z_L = M \left(\frac{\phi_o}{\pi}\right)^2 Z_o \frac{g^{long}}{I_o(gb)} C'_2 \tag{30}$$

with

$$C'_2 = \frac{r^2\ (\cos 2\Phi - \cos 2\theta) + jr \sin 2\Phi}{r \cos 2\Phi + \frac{1}{2} j(1+r^2)\sin 2\Phi} \tag{31}$$

As $\beta_W \to 1$, we get

$$C'_2 = \frac{j \sin 2\Phi}{\cos 2\Phi + j\frac{1+r^2}{2r} \sin 2\Phi} \tag{32}$$

$$\left|\frac{Z_L(r)}{Z_L(r=1)}\right| = \left[1 + \left(\frac{1-r^2}{2r}\right)^2 \sin^2 2\Phi\right]^{-1/2} . \tag{33}$$

Therefore Z_L can be decreased and stability improved by not matching Z_T and Z_o.

REFERENCES

1. Our derivation follows closely that of A.G. Ruggiero et al., ISR-RF-TH/69-7, CERN, March 1969.
2. R.E. Shafer, Fermilab UPC 133, July 1980.